MTH 203: Groups and Symmetry

Homework III

(Due 30/08)

Problems for submission

1. Establish the assertion in 1.1(ix)(b) of the Lesson Plan.
2. Establish the assertion in 2.1(ii) of the Lesson Plan.
3. Use question (1) to conclude that $\mathbb{R}^n$ is a group with respect to component-wise addition.
   (a) Show that every proper subgroup H that is also a subspace of $\mathbb{R}^2$ is a line through the origin.
   (b) Describe a typical coset of such a subgroup H.
4. Show that if a group has no proper subgroups, it must be cyclic of prime order.
5. Show that if in a group G, every non-identity element is of order 2, then G is abelian.

Problems for practice

1. Show that if a and b are integers with greatest $\gcd(a, b) = d$, then there exist integers x and y such that $ax + by = d$. (This is known as the Bézout's identity.)
2. Show that the following properties hold in a group G.
   (a) For $g \in G$, we have $(g^{-1})^{-1} = g$.
   (b) For $g' \in G$, we have
   $$g'G := \{g'g : g \in G\} = G.$$
   (c) For $g \in G$ and $a, b \in \mathbb{Z}$, we have
   $$g^a g^b = g^{a+b}.$$
   (d) For $g \in G$ and $a, b \in \mathbb{Z}$, we have
   $$g^{ab} = (g^a)^b = (g^b)^a.$$
   (e) For $g_1, g_2, \dots, g_k \in G$, we have
   $$(g_1 g_2 \dots g_k)^{-1} = g_k^{-1} g_{k-1}^{-1} \dots g_1.$$
3. Show that the following sets form groups.
   (a) $\mathrm{O}(n, F) = \{A \in \mathrm{GL}(n, F) : AA^{\intercal} = A^{\intercal}A = I_n\}$. (This is called the *orthogonal group in dimension n.*)
   (b) $\mathrm{SO}(n, F) = \{A \in \mathrm{O}(n, F) : \det(A) = 1\}$. (This is called the *special orthogonal group in dimension n.*)
4. Let G be an abelian group.
   (a) Show that $H = \{a \in G : a^2 = 1\}$ is a subgroup of G. Give an example of a nonabelian group in which H is not a subgroup.
   (b) Show that $H_n = \{a^n : a \in G\}$ is a subgroup of G.